

The phase structure of black D1/D5 (F/NS5) system in canonical ensemble

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Abstract

In this paper, we explore means which can be used to change qualitatively the phase structure of charged black systems. For this, we consider a system of black D1/D5 (or its S-dual F/NS5). We find that the delocalized charged black D-strings (F-strings) alone share the same phase structure as the charged black D5 branes (NS5-branes), having no van der Waals-Maxwell liquid-gas type. However, when the two are combined to form D1/D5 (F/NS5), the resulting phase diagram has been changed dramatically to a richer one, containing now the above liquid-gas type. The effect of adding the charged D-strings (F-strings) on the phase structure can also be effectively described as a slight increase of the transverse dimensions to the original D5 (NS5). This may be viewed as a connection between a brane charge and a fraction of spatial dimension at least in a thermodynamical sense.

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1 Introduction

Recent studies [1, 2] showed that a large part of the phase structure of a black hole in asymptotically anti-de Sitter (AdS) space [3, 4] is not unique to the AdS black hole, but actually shared universally by suitably stabilized black holes, say, in asymptotically flat space, even in the presence of a charge q . For example, a chargeless (suitably stabilized) black hole in asymptotically flat space can also undergo a Hawking-Page transition at certain temperature, now evaporating into a regular “hot flat space” instead of a regular “hot empty AdS space” as for an AdS black hole. Moreover, when $q \neq 0$, there exists also a critical charge q_c and for $q < q_c$, the phase diagram universally contains a van der Waals-Maxwell liquid-gas type phase structure along with a line of first-order phase transition terminating at a second-order critical point with a universal critical exponent for specific heat as⁴ $2/3$.

Unlike an AdS black hole for which the asymptotically AdS space itself acts as a reflecting box to stabilize the black hole, an isolated asymptotically flat black hole is thermodynamically unstable due to its Hawking radiation and needs to be stabilized first before its phase structure can be analyzed properly. For this, the standard practice is to place such a system inside a finite spherical cavity [5] with its surface temperature fixed. In other words, a thermodynamical ensemble is considered which can be either canonical or grand canonical, depending on whether the charge inside the cavity or the potential at the surface of the cavity is fixed [6]. Our focus in this paper is the canonical ensemble, i.e., the charge inside the cavity is fixed.

With the advent of AdS/CFT correspondence, the Hawking-Page transition for AdS black hole ‘evaporating’ into regular “hot empty AdS space” at certain temperature [7] corresponds to the confinement-deconfinement phase transition in large N gauge theory [8]. The very existence of the universal phase structure mentioned above strongly suggests that this universality is the result of the boundary condition rather than the exact details of asymptotical metrics which can be either flat, AdS or dS [1, 2]. The boundary condition realized in each case by the reflecting wall provides actually a confinement to the underlying system. This may suggest that the AdS holography is a result of such confinement rather than the detail properties of AdS space and it naturally leads to the speculation that a similar holography holds even in asymptotically flat space, as hinted in [1, 2]. If such a holography exists indeed, the natural and interesting questions are: what do the various thermal dynamical phase transitions correspond to in the field theory defined on the underlying holographic screen?

⁴Using the standard definition for this exponent, we have this value instead of $-2/3$ as given in [1, 9].

The same phase structure in the chargeless case was also found recently to be shared by the black p-branes in D-dimensional spacetime with the brane worldvolume dimensions $d = 1 + p$ in string/M theory and in the charged case the van der Waals-Maxwell liquid-gas type phase structure holds also true when $\tilde{d} = D - d - 2 > 2$ (note that $1 \leq \tilde{d} \leq 7$) [9, 10]. However, in the charged case we have a qualitatively different phase structure when $\tilde{d} < 2$ (there exists only the $\tilde{d} = 1$ case) which actually resembles that of chargeless case instead. The $\tilde{d} = 2$ case serves as a borderline in phase structure which distinguishes the $\tilde{d} = 1$ case from the $\tilde{d} > 2$ cases. For this case, there exists a ‘critical charge’ $q_c = 1/3$ and we have three subcases to consider, depending on the charge $q > q_c$, $q = q_c$ and $q < q_c$ (like the $\tilde{d} > 2$ cases), but we don’t actually have a critical point in the usual sense and for each subcase the phase structure looks more like that of the $\tilde{d} = 1$ case (see [9] for detail). So the qualitative phase structure in the charged case is actually determined by \tilde{d} , essentially the transverse dimensions (actually $\tilde{d} + 2$) to the branes.

For $D = 10$, the $\tilde{d} = D - d - 2 = 2$ gives $d = 6$ and the corresponding black p-brane are either D5 or its S-dual NS5-branes. The purpose of this paper is to explore means which can be used to change qualitatively the phase structure of charged D5 or its S-dual NS5-branes. We find that this can be achieved by adding delocalized charged D-strings (or F-strings) to the black D5 (or NS5) branes.

Specifically, we consider to add the charged D1 (or F-strings) to the D5 (or NS5) along one, say x^1 , and delocalized along the other four, say, x^2, x^3, x^4, x^5 of D5 (or NS5) worldvolume spatial directions. In other words, we consider the system of D1/D5 (or its S-dual F/NS5). At first look, one may attribute the qualitative change of phase structure of D5 (or NS5), which will be described later in this paper, to the added D1 (or F-strings) since one would naively think that for D-strings, the corresponding $\tilde{d} = D - 2 - d = 10 - 2 - 2 = 6 > 2$. This is actually not the case. For the delocalized charged black D-strings (or F-strings), their own phase structure (without the presence of D5 (or NS5)) is actually the same as that of charged black D5 (or NS5) as we will see in the later discussion. The surprise is that when both D5 (NS5) and the delocalized D-strings (F-strings) are present, the resulting phase structure is much richer and contains the van der Waals-Maxwell liquid-gas type, dramatically different from the one when only one type of branes are present. Concretely, we have now the reduced D5 (NS5) charge q_5 ($0 < q_5 < 1$) and the reduced D1 (F) charge q_1 ($0 < q_1 < 1$) and these two reduced charges form a two-dimensional region enclosed by a square with unit side. As a result, the present phase structure has a critical line instead of a critical point as in the charged black p-brane case with $\tilde{d} > 2$. This critical line divides the two-dimensional (q_1, q_5) charge region into two parts, with its one end approaching the point $(0, 1/3)$ and the other end approaching

the point $(1/3, 0)$. The point $(0, 1/3)$ is on the q_5 -axis (i.e., $q_1 = 0$) with $q_{5c} = 1/3$, the black D5 brane ‘critical charge’ mentioned earlier, without the presence of delocalized D1. Similarly, the point $(1/3, 0)$ is on the q_1 -axis with now $q_{1c} = 1/3$, the delocalized D1 ‘critical charge’, without the presence of D5 branes. Any point (q_1, q_5) in the part, defined as the region enclosed by the line interval $q_1 = 0, 0 \leq q_5 \leq 1/3$, the line interval $0 \leq q_1 \leq 1/3, q_5 = 0$ and the critical line, gives a van der Waals-Maxwell liquid-gas type phase structure, similar to the $q < q_c$ case as in black p-branes with $\tilde{d} > 2$, for which the equation of state can have three solutions if the cavity temperature is chosen properly. Any point (q_1, q_5) in the other part is similar to the $q > q_c$ case as in black p-branes with $\tilde{d} > 2$, for which the equation of state has a unique solution. Any given charge curve starting from one region, crossing the critical line and entering the other region will give a phase structure similar to the charged black p-branes with $\tilde{d} > 2$ as discussed in [9]. So adding the delocalized charged D-strings to D5 branes appears to increase effectively the transverse dimensions to the D5 branes from $\tilde{d} = 2$ to $\tilde{d} > 2$. Such an increase of \tilde{d} is found to be correlated, to certain extent, with that of the delocalized D-string (F-string) charge⁵ and the effective \tilde{d} is actually slightly greater than $\tilde{d} = 2$ when the delocalized D-string (F-string) charge is nonzero. So this indicates a possible connection between the brane charge and the effective spatial dimension, at least in the thermodynamical sense.

With one important exception, the other basic features of phase structure found for D1/D5 are also common to the two remaining systems D0/D4 and D2/D6 of the same type of $D(p - 4)/Dp$ for $p = 4, 6$, respectively. For either D0/D4 or D2/D6, this important exception is that the underlying phase structure of D4 or D6 by adding the delocalized D0 or D2 branes is, unlike the D1/D5 case, not changed qualitatively⁶. For this reason, we will present the main results for these two cases in the appendix.

This paper is organized as follows. In section 2, we present the configuration of charged black D1/D5 in Euclidean signature and derive the corresponding action from which we analyze the thermodynamical stability and obtain the inverse of local temperature for the purpose of understanding the underlying phase structure. The details of the underlying phase structure analysis are discussed in section 3. We explain what causes the dramatic change of the phase structure of black D5 branes when delocalized charged D-strings are added and discuss that the effect of adding these D1 on phase structure can be effectively described by a system of charged black p-branes with its \tilde{d} slightly greater than 2. We conclude this paper in section 4. The remaining two cases $D0/D4$ and $D2/D6$ of the

⁵Actually it is related to the critical charge of D-strings as we will discuss later.

⁶For D2/D6, with the addition of delocalized charged D2-branes, there is still some change of the underlying phase structure and it is now similar to that of D5-branes instead.

same type of $D(p-4)/Dp$ for $4 \leq p \leq 6$ are discussed in the Appendix.

2 The basic setup

Our focus in this paper is to see how the added delocalized charged D-strings to charged black D5-branes, in the sense described in the previous section, changes the phase structure of the D5-branes⁷ qualitatively. For this purpose, we focus only on the usual related phases of the black D1/D5 system. For simplicity, we will not consider here the corresponding bubble phase as found relevant in [11] in a similar fashion, which can become globally stable when certain conditions are met.

Let us consider the configuration of charged black D1/D5 [12], now expressed in Euclidean signature as

$$\begin{aligned}
ds^2 &= G_5^{-\frac{1}{4}} G_1^{-\frac{3}{4}} (f dt^2 + dx_1^2) + G_5^{-\frac{1}{4}} G_1^{\frac{1}{4}} \sum_{i=2}^5 dx_i^2 + G_5^{\frac{3}{4}} G_1^{\frac{1}{4}} \left(\frac{d\rho^2}{f} + \rho^2 d\Omega_3^2 \right) \\
A_{[2]} &= -ie^{-\phi_0/2} [\tanh \theta_1 - (1 - G_1^{-1}) \coth \theta_1] dt \wedge dx^1 \\
A_{[6]} &= -ie^{\phi_0/2} [\tanh \theta_5 - (1 - G_5^{-1}) \coth \theta_5] dt \wedge dx^1 \dots \wedge dx^5 \\
e^{2(\phi-\phi_0)} &= G_1/G_5,
\end{aligned} \tag{1}$$

where the metric is in Einstein frame, each form field is obtained following [6, 9] in such a way that the form field vanishes at the horizon so that it is well defined in the local inertial frame, ϕ_0 is the asymptotic value of the dilaton, $G_{1,5} = 1 + \frac{\rho_0^2 \sinh^2 \theta_{1,5}}{\rho^2}$ and $f = 1 - \frac{\rho_0^2}{\rho^2}$. The horizon is at $\rho = \rho_0$ with a curvature singularity behind at $\rho = 0$. Note that ρ_0, θ_1 and θ_5 are the parameters related to the mass, D1 charge and D5 charge of the system. For easily solving the parameter constraints considered later, we make the following radial coordinate transformation from ρ to r ,

$$\rho^2 = r^2 - r_-^2, \quad \rho_0^2 = r_+^2 - r_-^2, \tag{2}$$

with $r_+ \equiv \rho_0 \cosh \theta_5 \geq r_- \equiv \rho_0 \sinh \theta_5$. Now the horizon is at $r = r_+$ with the curvature singularity at $r = r_-$. We would like to stress that this particular choice of radial coordinate is just for convenience and the final results are independent of this. We can express

⁷In the text, for simplicity, we focus only on the system of D1/D5 with the understanding that when the effective string coupling becomes large, we should use its S-dual, i.e., F/NS5, instead, and the discussion remains exactly the same.

the above configuration as

$$\begin{aligned}
ds^2 &= \Delta_-^{1/4} G_1^{-3/4} \left(\frac{\Delta_+}{\Delta_-} dt^2 + dx_1^2 \right) + \Delta_-^{1/4} G_1^{1/4} \sum_{i=2}^5 dx_i^2 + \Delta_-^{1/4} G_1^{1/4} \left(\frac{dr^2}{\Delta_+ \Delta_-} + r^2 d\Omega_3^2 \right), \\
A_{[2]} &= -ie^{-\phi_0/2} [\tanh \theta_1 - (1 - G_1^{-1}) \coth \theta_1] dt \wedge dx^1 \\
A_{[6]} &= -ie^{\phi_0/2} [\tanh \theta_5 - (1 - \Delta_-) \coth \theta_5] dt \wedge dx^1 \dots \wedge dx^5 \\
e^{2(\phi - \phi_0)} &= G_1 \Delta_-,
\end{aligned} \tag{3}$$

where we have now

$$\Delta_{\pm} = 1 - \frac{r_{\pm}^2}{r^2}, \quad G_1 = 1 + \left(1 - \frac{\Delta_+}{\Delta_-} \right) \sinh^2 \theta_1. \tag{4}$$

Note that to have a finite Euclidean action for the D1/D5 system, the brane coordinates x^i with $i = 1, \dots, 5$ should be compact. For the above metric to be free from the conical singularity at the horizon, the time coordinate ‘t’ must be periodic with periodicity

$$\beta^* = 2\pi r_+ \cosh \theta_1, \tag{5}$$

the inverse of temperature of the black D1/D5 system at $r = \infty$. With this, the inverse of local temperature at a given r is

$$\beta = \Delta_+^{1/2} (\Delta_- G_1)^{-3/8} \beta^* = 2\pi \bar{r}_+ \Delta_+^{1/2} (\Delta_- G_1)^{-1/2} \cosh \theta_1, \tag{6}$$

where we have defined $\bar{r}_+ = r_+ (\Delta_- G_1)^{1/8}$ for the reason given below. Here r is merely a coordinate radius and the physical radius of the transverse 3-sphere is from the above metric as $\bar{r} = r (\Delta_- G_1)^{1/8}$ and the relevant local physical parameters $\bar{r}_{\pm} = r_{\pm} (\Delta_- G_1)^{1/8}$. Note that the expression of Δ_{\pm} remains the same even in terms of physical radial coordinate \bar{r} and parameters \bar{r}_{\pm} as

$$\Delta_{\pm} = 1 - \frac{r_{\pm}^2}{r^2} = 1 - \frac{\bar{r}_{\pm}^2}{\bar{r}^2}. \tag{7}$$

To study the equilibrium thermodynamics [13] in canonical ensemble, the allowed configuration (the black D1/D5 system or its extremal one) should be placed in a cavity with a fixed radius \bar{r}_B ($> \bar{r}_+$) for the reason as explained in the Introduction [5]. The other fixed quantities are the cavity temperature $1/\beta$, the physical periodicity of each x^i with $i = 1, \dots, 5$, the dilaton value $\bar{\phi}$ on the surface of the cavity (at $\bar{r} = \bar{r}_B$) and the charges/fluxes enclosed in the cavity \bar{Q}_p ($p = 1, 5$), respectively. In equilibrium, these fixed values are set equal to the corresponding ones of the allowed configuration enclosed in the cavity. For example, we set the charge

$$\bar{Q}_p = Q_p \equiv \frac{i}{\sqrt{2\kappa}} \int e^{-a(d)\phi} * F_{[d+1]}, \tag{8}$$

for $d = p + 1 = 2, 6$, respectively. In the above, $*$ denotes the Hodge duality and the field strength $F_{[d+1]} = dA_{[d]}$ with $A_{[d]}$ the corresponding form potential. With the potentials A_2, A_6 given in (3), we have

$$F_3 = -ie^{-\phi_0/2} \frac{r(r_+^2 - r_-^2) \sinh 2\theta_1}{[r^2 - r_-^2 + (r_+^2 - r_-^2) \sinh^2 \theta_1]^2} dr \wedge dt \wedge dx^1, \quad (9)$$

$$F_7 = -ie^{\phi_0/2} \frac{2r_-^2}{r^3} \coth \theta_5 dr \wedge dt \wedge dx^1 \wedge \cdots \wedge dx^5. \quad (10)$$

We therefore have the D-string charge and D5-brane charge per unit five-brane volume, respectively, as

$$Q_1 = \frac{\Omega_3 \bar{V}_4}{\sqrt{2}\kappa} e^{\bar{\phi}/2} \frac{(\bar{r}_+^2 - \bar{r}_-^2) \sinh 2\theta_1}{\Delta_- G_1}, \quad Q_5 = \frac{\Omega_3}{\sqrt{2}\kappa} e^{-\bar{\phi}/2} 2\bar{r}_+ \bar{r}_-, \quad (11)$$

where Ω_n denotes the volume of a unit n -sphere, κ is a constant with $1/(2\kappa^2)$ appearing in front of the Hilbert-Einstein action in canonical frame but containing no asymptotic string coupling g_s . In the above, we have expressed all the quantities in terms of their fixed (or physical) correspondences, for example, we have replaced the asymptotical string coupling g_s in terms of the fixed effective string coupling on the cavity via $e^{\bar{\phi}} = e^{\phi(\bar{r}_B)} \equiv g_s(\Delta_- G_1)^{1/2}$. The above physical volume $\bar{V}_4 = (\Delta_- G_1)^{1/2} V_4^*$ with the coordinate volume $V_4^* \equiv \int dx^2 dx^3 dx^4 dx^5$. In canonical ensemble, it is the Helmholtz free energy which determines the stability of equilibrium states and is related to the Euclidean action by $F = I_E/\beta$ in the leading order approximation. So, in order to understand the phase structure we will evaluate the action for the black D1/D5.

The procedure for evaluating the Euclidean action of black p -branes was given in detail in [9] following the standard technique. The generalization to the present case is straightforward, though the computation is a bit lengthy, by considering one more piece contribution from the form field strength F_3 and its potential A_2 (or F_{p-2} and its potential A_{p-3} in general for $p \geq 4$) in addition to that from the usual F_7 and its potential A_6 (F_{p+2} and its potential A_{p+1}) in the action. After a bit lengthy computations, we have the Euclidean action for the black D1/D5 system as,

$$I_E = -\frac{\beta \bar{V}_5 \Omega_3}{\kappa^2} \bar{r}_B^2 \left[2 \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} + (\Delta_+ \Delta_-)^{1/2} - 3 - \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} (1 - G_1^{-1}) \right] \\ - \frac{2\pi \bar{r}_+ \bar{V}_5 \Omega_3}{\kappa^2} \bar{r}_B^2 \left(1 - \frac{\Delta_+}{\Delta_-} \right) \left[1 + \frac{1 - G_1^{-1}}{\frac{\Delta_-}{\Delta_+} - 1} \right]^{1/2}, \quad (12)$$

with Δ_{\pm} , G_1 taking their respective value at $\bar{r} = \bar{r}_B$. Since the Helmholtz free energy is given as $F = E - TS$, so we have $I_E = \beta E - S$, where E is the internal energy and

S is the entropy of the black D1/D5. Thus we identify the internal energy of the black D1/D5 on dividing the first term in (12) by β and can be checked to match the ADM mass per unit 5-brane volume of the system as $\bar{r}_B \rightarrow \infty$. The second term in (12) is the entropy of the system which can be checked directly via $S = A/4G$ by computing A , the physical area transverse to the radial direction at $r = r_+$ from the metric (3), with $8\pi G = \kappa^2$. In (12), the 5-brane physical volume $\bar{V}_5 = G_1^{1/8} \Delta_-^{5/8} V_5^*$ with the coordinate 5-brane volume $V_5^* = \int dx^1 \cdots dx^5$. Note also that in obtaining action (12), we have used the second expression in (4) to replace the parameter θ_1 in terms of Δ_{\pm} and G_1 via (assuming $\theta_1 \geq 0$)

$$\sinh \theta_1 = \left(\frac{G_1 - 1}{1 - \frac{\Delta_+}{\Delta_-}} \right)^{1/2}. \quad (13)$$

In action (12), it appears that at this stage we have three variables \bar{r}_+ , \bar{r}_- and G_1 since \bar{V}_5, β and \bar{r}_B are the fixed boundary data (Note that Δ_{\pm} is a function of \bar{r}_{\pm} , respectively). This is actually not the case since we have not used the two charge expressions given in (11) for which the two charges are set to fixed in the canonical ensemble. So we are left with only one variable and it is actually \bar{r}_+ . Given the fixed boundary data, we can define two new fixed charge quantities \tilde{Q}_1 and \tilde{Q}_5 , respectively, from (11) as

$$\begin{aligned} \tilde{Q}_1 &= \frac{\sqrt{2\kappa}\bar{Q}_1 e^{-\bar{\phi}/2}}{2\bar{V}_4 \bar{r}_B^2 \Omega_3} = (1 - G_1^{-1})^{\frac{1}{2}} \left[1 - \frac{\Delta_+}{\Delta_-} + \frac{\Delta_+}{\Delta_-} (1 - G_1^{-1}) \right]^{\frac{1}{2}} < 1 \\ \tilde{Q}_5^2 &= \frac{\sqrt{2\kappa}\bar{Q}_5 e^{\bar{\phi}/2}}{2\Omega} = \bar{r}_+ \bar{r}_-, \end{aligned} \quad (14)$$

where we have set $Q_1 = \bar{Q}_1$ and $Q_5 = \bar{Q}_5$ and in the second equality of the first equation we have used (13) to express θ_1 in terms of Δ_{\pm} and G_1 . Note that the fixed \tilde{Q}_1 is dimensionless while the \tilde{Q}_5 has a dimension of length square. From the second equation, we have $\bar{r}_- = \tilde{Q}_5^2 / \bar{r}_+$ and so \bar{r}_- is expressed in terms of \bar{r}_+ (therefore Δ_- is also expressed in \bar{r}_+). From the first equation, we can solve a quadratic equation to find the relevant root

$$1 - G_1^{-1} = \frac{1}{2} \left[\sqrt{\left(\frac{\Delta_-}{\Delta_+} - 1 \right)^2 + 4\tilde{Q}_1^2 \frac{\Delta_-}{\Delta_+}} - \left(\frac{\Delta_-}{\Delta_+} - 1 \right) \right], \quad (15)$$

where we have used the fact $1 - G_1^{-1} > 0$. With these, we can see that the action (12) is a function of variable \bar{r}_+ only.

For simplicity, we define the relevant reduced action

$$\begin{aligned}
\bar{I}_E &\equiv \frac{\kappa^2 I_E}{2\pi \tilde{V}_5 \Omega_3 \bar{r}_B^3} \\
&= -\frac{\beta}{2\pi \bar{r}_B} \left[2 \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} + (\Delta_+ \Delta_-)^{1/2} - 3 - \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} (1 - G_1^{-1}) \right] \\
&\quad - \frac{\bar{r}_+}{\bar{r}_B} \left(1 - \frac{\Delta_+}{\Delta_-} \right) \left[1 + \frac{1 - G_1^{-1}}{\frac{\Delta_-}{\Delta_+} - 1} \right]^{1/2}.
\end{aligned} \tag{16}$$

We also define the following reduced quantities

$$x \equiv \left(\frac{\bar{r}_+}{\bar{r}_B} \right)^2 < 1, \quad \bar{b} \equiv \frac{\beta}{4\pi \bar{r}_B}, \quad q_5 = \left(\frac{\tilde{Q}_5}{\bar{r}_B} \right)^2 < x, \quad q_1 = \tilde{Q}_1. \tag{17}$$

Note that $0 < q_5 < 1$ (since $x < 1$) and $0 < q_1 < 1$ (since $\tilde{Q}_1 < 1$ from the first equation in (14)). With these, we have

$$\Delta_+ = 1 - \frac{\bar{r}_+^2}{\bar{r}_B^2} = 1 - x, \quad \Delta_- = 1 - \frac{\bar{r}_-^2}{\bar{r}_B^2} = 1 - \frac{q_5^2}{x}, \tag{18}$$

where for Δ_- we have used $\bar{r}_- = \tilde{Q}_5^2 / \bar{r}_+$. In terms of these reduced quantities, we have the reduced action (16) as

$$\begin{aligned}
\bar{I}_E &= -2\bar{b} \left[2 \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} + (\Delta_+ \Delta_-)^{1/2} - 3 - \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} (1 - G_1^{-1}) \right] \\
&\quad - x^{1/2} \left(1 - \frac{\Delta_+}{\Delta_-} \right) \left[1 + \frac{1 - G_1^{-1}}{\frac{\Delta_-}{\Delta_+} - 1} \right]^{1/2},
\end{aligned} \tag{19}$$

where

$$\frac{\Delta_+}{\Delta_-} = \left(\frac{\Delta_-}{\Delta_+} \right)^{-1} = \frac{1 - x}{1 - \frac{q_5^2}{x}}, \tag{20}$$

and $1 - G_1^{-1}$ is given in terms of Δ_- / Δ_+ via (15) with now $\tilde{Q}_1 = q_1$. So the above reduced action is only a function of variable x with a few fixed parameters \bar{b}, q_1, q_5 .

The thermal equilibrium of black D1/D5 with the cavity can be determined, after a rather lengthy calculation, from

$$\frac{d\bar{I}_E}{dx} \sim (\bar{b} - b_{q_1, q_5}(x)) \Rightarrow 0, \tag{21}$$

to be

$$\bar{b} = b_{q_1, q_5}(\bar{x}). \tag{22}$$

The local minima of \bar{I}_E , therefore the local stability, require

$$\left. \frac{d^2 \bar{I}_E}{dx^2} \right|_{x=\bar{x}} \sim -\frac{db_{q_1, q_5}(\bar{x})}{d\bar{x}} > 0, \quad (23)$$

which in turn needs the negative slope of $b_{q_1, q_5}(\bar{x})$ at the equilibrium. In the above, the inverse of the reduced temperature function

$$b_{q_1, q_5}(x) = \frac{1}{2} x^{1/2} \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} \left[1 + \frac{1 - G_1^{-1}}{\frac{\Delta_-}{\Delta_+} - 1} \right]^{1/2}, \quad (24)$$

which agrees with the inverse of local temperature (6) divided by $4\pi\bar{r}$ at $\bar{r} = \bar{r}_B$, therefore a consistent check. This function is the basis for the analysis of the underlying phase structure which we will perform in the following section.

3 The analysis of phase structure

Before we discuss the surprise mentioned in the Introduction, we first examine a few things implied in the function $b_{q_1, q_5}(x)$ derived in the previous section. First, when we set the delocalized D-string charge $q_1 = 0$, we end up with the inverse of the reduced temperature function $b_{q_5}(x)$, as expected, for the black 5-branes given in [9]

$$b_{0, q_5}(x) = b_{q_5}(x) = \frac{1}{2} x^{1/2} \left(\frac{1 - x}{1 - \frac{q_5^2}{x}} \right)^{1/2}. \quad (25)$$

Note here $q_5 < x < 1$. Let us examine what happens if we set the 5-brane charge $q_5 = 0$. We now have from (24)

$$b_{q_1, 0}(x) = \frac{1}{2} (1 - x)^{1/2} \left(\frac{x + \sqrt{x^2 + 4q_1^2(1 - x)}}{2} \right)^{1/2}, \quad (26)$$

where we have used $\Delta_- = 1$ and the explicit expressions for Δ_+ in (18) and $1 - G_1^{-1}$ as given in (15), respectively. Note now $0 < x < 1$ and it appears that the above $b_{q_1, 0}(x)$ looks quite different from $b_{0, q_5}(x)$ given in (25). This apparent difference is actually due to the improper use of the variable x for the present case. As we stress in the previous section, we make a special choice of the radial coordinate as given in (2) as well as r_{\pm} so that the charge constraints given in (14) are simplified greatly and can be solved explicitly. While this is good for convenience, it is not essential and our choice there prefers actually to the 5-branes. When $q_5 = 0$, we have the range of variable x in $0 < x < 1$ which

merely reflects the 5-branes being chargeless, certainly not a good one in describing the delocalized strings. We expect a relevant variable y in the range $q_1 < y < 1$ and this can be achieved via⁸

$$y = \frac{x + \sqrt{x^2 + 4q_1^2(1-x)}}{2}, \quad (27)$$

from which we solve for $1-x$ as

$$1-x = \frac{1-y}{1-\frac{q_1^2}{y}}. \quad (28)$$

Using the y variable in (26), we have

$$b_{q_1,0}(y) = \frac{1}{2}y^{1/2} \left(\frac{1-y}{1-\frac{q_1^2}{y}} \right)^{1/2}, \quad (29)$$

which is identical in form to $b_{0,q_5}(x)$ given in (25) but now with variable y and its range $q_1 < y < 1$. So the delocalized charged black D-strings, unlike the localized charge black D-strings, have the same phase structure as the black D5-branes, described in detail in [9]. In other words, so long the phase structure is concerned, the delocalized charged black D-strings look no different from the black D5-branes, which is completely determined by the function $b_{q_1,0}(y)$ or $b_{0,q_5}(x)$.

Given what has been said above, one might naively conclude that when both the 5-branes and the delocalized strings are present, the underlying phase structure would remain the same as that when either are present. This is actually not the case, a surprise, as we will see in what follows. For this, let us examine the full expression $b_{q_1,q_5}(x)$ (24) when both q_1 and q_5 are non-zero. The above discussion indicates already that the variable x is not a good one since it favors 5-branes over the delocalized strings and we would like to have one which can give a symmetric representation between them. This can be achieved via the following⁹

$$f = \frac{\Delta_+}{\Delta_-} = \frac{1-x}{1-\frac{q_5^2}{x}} < 1, \quad (30)$$

⁸One can show directly that in analogous to the variable x favoring D5-branes, the variable y is indeed the one favoring the delocalized D-strings in a similar fashion. For this, in analogous to r and r_{\pm} in (2), we define here $\rho^2 = \tilde{r}^2 - \tilde{r}_-^2$, $\rho_0^2 = \tilde{r}_+^2 - \tilde{r}_-^2$ with $\tilde{r}_+ = \rho_0 \cosh \theta_1$ and $\tilde{r}_- = \rho_0 \sinh \theta_1$. So we expect $y = (\tilde{r}_+/\tilde{r}_B)^2 = \rho_0^2 \cosh^2 \theta_1 / (\rho_B^2 + \rho_0^2 \sinh^2 \theta_1) = x \cosh^2 \theta_1 / (1 + x \sinh^2 \theta_1)$ where $x = (\rho_0/\rho_B)^2$ when $q_5 = 0$. When $q_5 = 0$, $r_- = 0$ and from the first equation in (11) and the definition for q_1 given in (14) and (17), we have $q_1 = x \cosh \theta_1 \sinh \theta_1 / (1 + x \sinh^2 \theta_1)$. Combining these two equations, we have $y^2 - xy - q_1^2(1-x) = 0$ whose proper solution is nothing but (27) with $q_1 < y < 1$.

⁹This variable f , in terms of the original variable ρ (and ρ_0) given in (1), is $f = 1 - \rho_0^2/\rho_B^2$.

from which we can solve to give (noting $q_5 < x < 1$)

$$x = \frac{1 - f + \sqrt{(1 - f)^2 + 4q_5^2 f}}{2}. \quad (31)$$

In the above, note that $x = q_5$ corresponds to $f = 1$ while $x = 1$ to $f = 0$ and the range for f is now $0 < f < 1$. With the new variable f , we have the reduced action (19)

$$\begin{aligned} \bar{I}_E = & -2\bar{b} \left[2f^{\frac{1}{2}} + f^{-\frac{1}{2}} - 3 - \frac{\sqrt{(1-f)^2 + 4q_1^2 f} + \sqrt{(1-f)^2 + 4q_5^2 f}}{2f^{1/2}} \right] \\ & -(1-f)^{\frac{1}{2}} \left[\frac{1-f + \sqrt{(1-f)^2 + 4q_1^2 f}}{2} \right]^{\frac{1}{2}} \left[\frac{1-f + \sqrt{(1-f)^2 + 4q_5^2 f}}{2} \right]^{\frac{1}{2}}, \end{aligned} \quad (32)$$

and the function (24)

$$b_{q_1, q_5}(f) = \frac{1}{4} \left[\frac{f \left(1 - f + \sqrt{(1-f)^2 + 4q_1^2 f} \right) \left(1 - f + \sqrt{(1-f)^2 + 4q_5^2 f} \right)}{1 - f} \right]^{1/2}, \quad (33)$$

both of which do reflect the symmetry between q_1 and q_5 . The equation of state (22) becomes now

$$\bar{b} = b_{q_1, q_5}(\bar{f}). \quad (34)$$

Let us examine the behavior of $b_{q_1, q_5}(f)$ which will determine the phase structure of D1/D5 system. Note that $b_{q_1, q_5}(f) > 0$ for $0 < f < 1$ and when neither charge is zero, we have¹⁰

$$b_{q_1, q_2}(f \rightarrow 1) \rightarrow \infty, \quad b_{q_1, q_5}(f \rightarrow 0) \rightarrow 0. \quad (35)$$

The corresponding function for black p-branes with $\tilde{d} > 2$ has also the same behavior[9], critical to the underlying phase structure. So we expect here the similar phase structure, namely, the existence of van der Waals-Maxwell liquid-gas type phase structure¹¹. Unlike

¹⁰This same behavior can be seen from the original $b_{q_1, q_5}(x)$ (24) using variable x : $b_{q_1, q_5}(x \rightarrow q_5) \rightarrow \infty$, noting $\Delta_+/\Delta_- \rightarrow 1$ when $x \rightarrow q_5$, and $b_{q_1, q_5}(x \rightarrow 1) \rightarrow 0$.

¹¹Note that from (31), we have

$$2 dx = - \left[1 + \frac{1 - f - 2q_5^2}{\sqrt{(1-f)^2 + 4q_5^2 f}} \right] df, \quad (36)$$

and since $\sqrt{(1-f)^2 + 4q_5^2 f} > |1 - f - 2q_5^2|$, we have $dx \sim -df$ and hence the negative slope of $b_{q_1, q_5}(x)$, determining the local minima of the free energy (23) at an equilibrium point, corresponds to the positive slope of $b_{q_1, q_5}(f)$. So in what follows, we seek the positive slope of $b_{q_1, q_5}(f)$ for the local minima of free energy at an equilibrium.

the black p-brane case, we have here a critical line instead, which can be determined from the following two conditions:

$$\frac{db_{q_1, q_5}(f)}{df} = 0, \quad \frac{d^2 b_{q_1, q_5}(f)}{df^2} = 0. \quad (37)$$

From the above, we have

$$\frac{1}{A} + \frac{1}{B} = \frac{2(2-f)}{1+f}, \quad \frac{1}{A^3} + \frac{1}{B^3} = \frac{2(2-3f+6f^2-f^3)}{(1+f)^3}, \quad (38)$$

where

$$A = \sqrt{1 + 4q_1^2 \frac{f}{(1-f)^2}}, \quad B = \sqrt{1 + 4q_5^2 \frac{f}{(1-f)^2}}. \quad (39)$$

Unlike the black p-brane case[9], we have here three critical quantities q_{1c}, q_{5c}, f_c to be determined but with only two equations in (38). This must imply that we will have a critical line rather than a critical point as for the black p-brane case. The critical charges q_{5c} and q_{1c} , representing a line, can be solved in terms of the critical parameter f_c as

$$\begin{aligned} q_{5c} &= \sqrt{\frac{(1+f_c)^2 \left[\sqrt{(2-f_c)(3f_c-2)} \pm (2-f_c)^2 \right]^2}{4f_c(10-5f_c+f_c^2)^2}} - \frac{(1-f_c)^2}{4f_c}, \\ q_{1c} &= \sqrt{\frac{(1+f_c)^2 \left[\sqrt{(2-f_c)(3f_c-2)} \mp (2-f_c)^2 \right]^2}{4f_c(10-5f_c+f_c^2)^2}} - \frac{(1-f_c)^2}{4f_c}. \end{aligned} \quad (40)$$

In the above, the existence of the critical line actually requires $f_c \geq 2/3$. So the range for f_c is $2/3 \leq f_c < 1$.

Given $0 < q_1 < 1$ and $0 < q_5 < 1$, the two reduced charges q_1 and q_5 form a two dimensional region enclosed by a square with a unit side as shown in Fig. 1. The critical line divides the region into two parts, the upper part and the lower one. Given what we know about the phase structure of black 5-branes (corresponding to the q_5 -axis with $q_1 = 0, 0 < q_5 < 1$ and $q_{5c} = 1/3$) or the delocalized D-strings (corresponding to the q_1 -axis with $0 < q_1 < 1, q_5 = 0$ and $q_{1c} = 1/3$), we should have the upper part of the region to give rise to the phase structure similar to the $q > q_c$ case for black p-branes when $\tilde{d} > 2$ and the lower part to the $q < q_c$ case containing the van der Waals-Maxwell liquid-gas type structure. Before we examine this further, we would like to discuss a bit more about the charge parameter-space structure.

Given that the delocalized charged black strings and the charged black 5-branes each share the same phase structure, one expects that the charge parameter-space should be

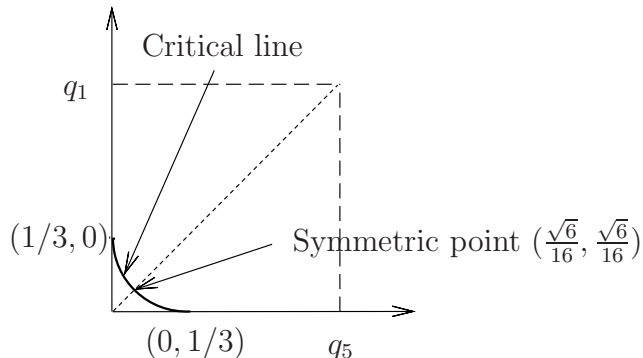


Figure 1: The reduced charges q_1 and q_5 form a two-dimensional region bounded by a square with a unit side and the critical line divides the region with the upper part giving rise to the phase similar to the $q > q_c$ case for black p-branes when $\tilde{d} > 2$ while the lower part to the $q < q_c$ case containing a van der Waals-Maxwell liquid-gas type structure.

symmetric with respect to q_1 and q_5 for the phase structure. In other words, the dotted diagonal line of the square connecting the origin and the opposite corner in Fig. 1 divides the parameter space into two equivalent regions. This is further support by the critical charges given in (40) where q_{1c} and q_{5c} do appear symmetric and have a symmetric point at the intersection point $(\sqrt{6}/16, \sqrt{6}/16)$ between the diagonal line and the critical line (also at the middle of the critical line). For this reason and later for comparison with black 5-branes, we will focus, from now on, on the lower half of the region including the diagonal line, i.e. the region enclosed by the diagonal line, the q_5 -axis and the $q_5 = 1$ line plus the diagonal line itself. For the critical line, we will choose in (40) the plus sign for q_{5c} and minus sign for q_{1c} with $2/3 \leq f_c < 1$ (Note that the symmetric critical point occurs at $f_c = 2/3$). For convenience, we denote the region having the phase structure similar to the $q > q_c$ case for black p-branes when $\tilde{d} > 2$ as “upper region” and the one containing the van der Waals-Maxwell liquid-gas type as “lower region”. We now come back to give more evidence in support of our above claims about the phase structure in “upper region” and in “lower region” of the charge space. We cannot give a complete analysis for the present case, in a spirit similar to that in [9], due to the complexity of the first equation in (37). However, we know here already the critical line and so either “upper region” or “lower region” must correspond to one or the other, but not both, phase structure mentioned above. So selecting a few sampling points in each region will be sufficient to confirm our above claims. For this, we consider three points

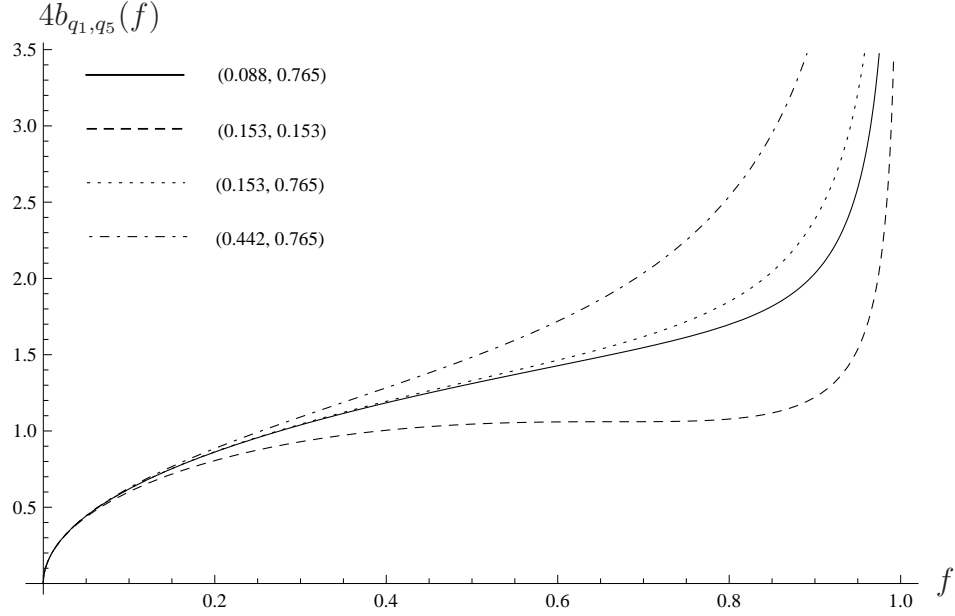


Figure 2: The expected behavior of $b_{q_1, q_5}(f)$ vs f for three selected pairs of (q_1, q_5) values in the region enclosed by the upper part of diagonal line and the lower half of critical line and the $q_5 = 1$ line as shown in Fig. 1. Note that the pair of $(0.153, 0.153)$ corresponds to the symmetric point on the critical line.

$(q_1, q_5) = (0.088, 0.765), (0.153, 0.765), (0.442, 0.765)$ in “upper region” and one critical point $(q_1, q_5) = (0.153, 0.153)$, the symmetric point on the critical line. As shown in Fig. 2, there doesn't exist any extremum of $b_{q_1, q_5}(f)$ for any of three points selected and for each given \bar{b} , there will be a unique solution of $\bar{b} = b_{q_1, q_5}(f)$. Moreover, given the behavior of $b_{q_1, q_5}(f)$ for $f \rightarrow 0$ and $\rightarrow 1$ in (35), $b_{q_1, q_5}(f)$ increases monotonically for $0 < f < 1$ and its slope is hence positive. So the corresponding free energy is a local minimum, therefore stable. The symmetric critical point $(0.153, 0.153)$ in Fig. 2 does appear to be flat around the point $f = 2/3$ which gives the critical point. Let us also examine three sampling points $(q_1, q_5) = (0.031, 0.063), (0.044, 0.088), (0.054, 0.099)$ but now in “lower region”. As shown in Fig. 3, each gives rise to a maximum and a minimum of $b_{q_1, q_5}(f)$ for $0 < f < 1$, as expected. When \bar{b} falls between the minimum and maximum, the equation of state $\bar{b} = b_{q_1, q_5}(f)$ will have three solutions $f_1 < f_2 < f_3$ but only at the smallest f_1 or at the largest f_3 , the slope of $b_{q_1, q_5}(f)$ is positive and the corresponding free energy has a local minimum, therefore locally stable. While at the middle f_2 , the slope of this function is negative and the free energy has a maximum there, therefore unstable. For a given pair of charge (q_1, q_5) , when we change \bar{b} between the maximum and

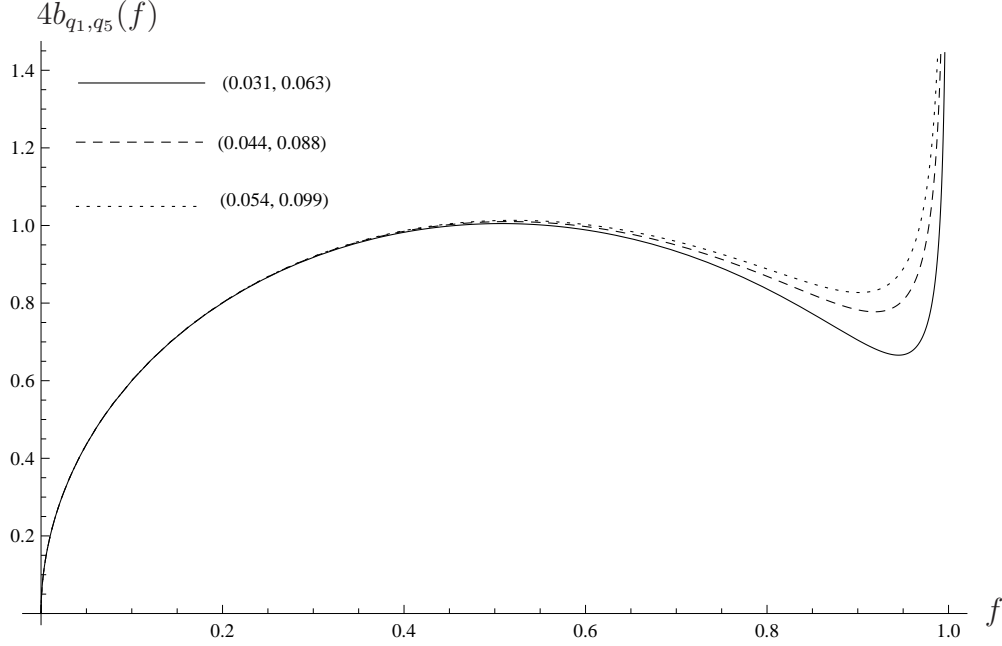


Figure 3: The expected behavior of $b_{q_1, q_5}(f)$ vs f for three selected pairs of (q_1, q_5) values in the region enclosed by the lower part of diagonal line and the lower half of critical line and the q_5 -axis as shown in Fig. 1.

the minimum, each of these \bar{b} will give two locally stable phases but only the one with the lowest free energy is more stable and the other one is merely meta-stable. Among these \bar{b} , there exists a special \bar{b}_t which is completely determined by these two charges and the two locally stable phases have the same free energy, therefore the two can co-exist and there can be a phase transition between the two. Since f is related to the horizon size x via (31) and a larger f corresponds to a smaller x , so f_1 corresponds to a larger horizon size while f_3 corresponds to a smaller horizon size. Therefore such a phase transition involves a change of horizon size, implying a change of entropy of the underlying system. So we have this phase transition a first-order one. The \bar{b}_t is just the inverse of the corresponding phase transition reduced temperature. Following [9], when the temperature is lower than the transition one, i.e., $\bar{b} > \bar{b}_t$, the large f_3 (therefore corresponding to a small horizon size) phase will have a lower free energy, therefore more stable. Otherwise, the small f_1 phase will have a lower free energy, therefore more stable. So we also have here a van der Waals-Maxwell liquid-gas phase structure as anticipated. Here the discussion is similar to the $q < q_c$ case of black p-branes when $\tilde{d} > 2$ given in [9]. In drawing analogy to the liquid-gas phase diagram, here the small stable D1/D5 system corresponds to the liquid

phase while the large one to the gas phase.

Given what has been said, an analysis for a special case of $q_1 = q_5$, in a spirit similar to that in [9], is still possible for the phase structure. This will lend further support to our above claims. In other words, we now consider the diagonal line of the charge square in Fig. 1. From the first equation in (37) or (38), we have the following equation, which determines the position of the extremality of $b_{q,q}(f)$, as

$$(6 - 4q^2)f^3 - (15 - 16q^2)f^2 + (12 - 16q^2)f - 3 = 0, \quad (41)$$

where for simplicity we have set $q_1 = q_5 = q$. This is a cubic equation and has three roots. We will discuss the properties of these three roots, following the analysis of [9], to determine the behavior of $b_{q,q}(f)$, therefore the phase structure. For this, we have the discriminant of the above cubic equation as

$$\Delta(q) = 144q^2(q - 1)(q + 1)(128q^2 - 3). \quad (42)$$

Given $0 < q < 1$, we know that the discriminant can only vanish if $q = q_c = \sqrt{6}/16$, which is nothing but the expected symmetric (middle) point on the critical line for which the two positive roots coincide at $f_c = 2/3$. The third root is actually negative, therefore not in the region of our interest, since the sign of the product of three roots is given by that of the ratio of the coefficient of f^3 to the constant term in (41), which is negative.

When $1 > q > q_c$, we have $\Delta(q) < 0$ and so we have now a pair of complex conjugate roots and one negative root. None of them are in the region of our interest. In other words, $b_{q,q}(f)$ doesn't have any extremality in the region $0 < f < 1$ and so $b_{q,q}(f)$ must increase monotonically in this region given its behavior for $f \rightarrow 0$ and $f \rightarrow 1$ in (35). This further implies its slope being positive and a unique solution from the equation of state $\bar{b} = b_{q,q}(\bar{f})$. So the free energy at each equilibrium point $f = \bar{f}$ has a minimum, therefore stable. This is nothing but the behavior of the $q > q_c$ case of black p-branes when $\tilde{d} > 2$ considered in [9].

When $q < q_c$, we have $\Delta(q) > 0$ and so the three roots are all real. Note that the left side of equation (41) takes a value of -3 at $f = 0$ and a value of $-4q^2$ at $f = 1$, both being negative at the two ends of $0 < f < 1$, therefore the number of roots, if they exist at all in the region of $0 < f < 1$, must be even (two or none). Given the fact about the existence of critical point $f_c = 2/3$, which results from the coincidence of two positive roots when $q \rightarrow q_c$, this implies therefore the existence of two positive roots indeed in the region of $0 < f < 1$. Here the analysis goes parallel to that given in [9] for $q < q_c$ case of black p-branes when $\tilde{d} > 2$ and so we expect that the underlying phase structure contains also the van der Waals-Maxwell liquid-gas type.

As in [9] for black p-branes with $\tilde{d} > 2$, the extremal case (corresponding to taking $f = 1$) with the same boundary data has always larger free energy than the non-extremal case, therefore can never be a stable thermodynamical state in canonical ensemble¹².

In summary, with the addition of delocalized charged D-strings, the phase structure of charged black D5-branes with $\tilde{d} = 2$ as described in [9] has been qualitatively changed to the one similar to that of black p-branes with $\tilde{d} > 2$, a van der Waals-Maxwell liquid-gas type, even though the delocalized charged black D-strings alone share the same phase structure as the original charged black D5-branes, without such a type phase structure. The present phase structure is actually even more richer since we have now a critical line instead of a critical point which separates a two-dimensional charge square into two regions with each giving rise to a different phase behavior as described earlier. Each point on the critical line represents a second-order phase transition. Any continuous curve crossing the critical line in the two-dimensional charge square gives rise to a phase structure like that of black p-branes with $\tilde{d} > 2$, having a first-order phase transition line which terminates at a second-order phase transition point, corresponding to the crossing point between this charge curve and the critical line.

At one hand, the delocalized charged black D-strings share the same phase structure as the charged black D5-branes [9] with $\tilde{d} = 2$. This seemingly indicates that the delocalized charged black D-strings behave, at least in terms of thermodynamics, like the charged

¹²With the same boundary data, the extremal case has its free energy $\bar{F}_{\text{extremal}} = \bar{I}_E^{\text{extremal}}/\bar{b} = 2(q_1 + q_5)$ following the discussion given in [11] while the non-extremal one has its on-shell free energy $\bar{F}_{\text{non-extremal}}(\bar{f}) = -2\bar{f}^{-1/2}[\bar{f} - 3\bar{f}^{1/2} + 2 - (\sqrt{(1-\bar{f})^2 + 4q_1^2\bar{f}} + \sqrt{(1-\bar{f})^2 + 4q_5^2\bar{f}})/2]$ where we have used the equation of state (34) and \bar{f} is a solution from (34) for each given \bar{b} . We expect in general $0 < \bar{f} < 1$ but not every its value, when in “lower region”, gives a stable phase. Note that $\bar{F}_{\text{non-extremal}}(\bar{f} = 0) = -\infty$ and $\bar{F}_{\text{non-extremal}}(\bar{f} = 1) = 2(q_1 + q_5)$ at its two limiting ends. We would like to point out that though both $\bar{F}_{\text{non-extremal}}(\bar{f} = 1)$ and $\bar{F}_{\text{extremal}}$ have the same value $2(q_1 + q_5)$ but the extremal one has this value, independent of \bar{f} or \bar{b} , following [14, 15, 16, 17]. One can check directly for the on-shell free energy $d\bar{F}_{\text{non-extremal}}/d\bar{f} \sim db_{q_1, q_5}(\bar{f})/d\bar{f}$. Note that $\bar{F}_{\text{non-extremal}}(\bar{f})$ has its value as a local minimum at $f = \bar{f}$ and therefore $db_{q_1, q_5}(\bar{f})/d\bar{f} > 0$. This implies that the on-shell free energy has its $d\bar{F}_{\text{non-extremal}}/d\bar{f} > 0$ for locally stable phase. In other words, except for at the phase transition point if relevant at all, the on-shell free energy for stable phase is a monotonically increasing function of variable \bar{f} and so this further implies that the end value of $\bar{F}_{\text{non-extremal}}(\bar{f} = 1) = 2(q_1 + q_5)$ is the largest one. Therefore the extremal one has always the largest free energy given what has been said above. The similar line of argument was also employed in [11] for the case of black p-branes with $\tilde{d} > 2$. Note that for the present extremal case, the total free energy is simply the sum of the free energies due to the delocalized strings and D5-branes while this is not true for the non-extremal case. This lowered free energy for non-extremal case should be due to the attractive interaction between the delocalized strings and D5-branes while for extremal case, the delocalized strings don’t interact with the D5-branes since their brane dimensions differ by four. We will elaborate this point later in this section.

black D5-branes. Therefore one would naively expect that when the two are combined to form black D1/D5 system, its phase behavior remains the same as before. But on the other hand, to our surprise, the resulting phase structure is actually completely different as revealed in this study. As hinted already in footnote 12 given previously, this is due to the attractive interaction between the black D-strings and the black D5-branes, which is quite different from that between black D5-branes. We will elaborate this further in what follows.

When both D1 and D5 are extremal or BPS, it is well-known that there exist no interaction between any two of these branes, whether they are both D1 or both D5 or one D1 and the other D5. As given in footnote 12, the total free-energy of extremal D1/D5 system is just the sum of constituent brane contributions and the reason behind is just the mentioned no interactions among these branes. For charged black D-strings or/and D5-branes, the story is different. For either type of branes, the net interaction is attractive since the attractive gravitational one due to brane masses overtakes the repulsive one due to their R-R charges. For each given type of branes, the corresponding black configuration remains the same in form and so the form of its Euclidean action remains also the same, independent of its charge. This can be easily seen, for example, from (32) when we set either q_1 or q_5 to zero. When both q_1 and q_5 are non-zero, the resulting form is completely different from that when only one of charges is non-zero. This is due to the following reasons. D-string R-R charges don't interact with D5 ones and the interaction between them are due to their masses. This attractive interaction is different from that between D-strings or between D5-branes and it is not additive.

The above understanding is based on the possible interactions which appear in the system. In terms of thermodynamics, the understanding is a bit different and the non-additive contributions appear in both the internal energy and the entropy of the underlying system which can be seen from, for example, the reduced action (32). Let us examine this action in a bit detail. First, if we set either q_1 or q_5 zero in the action, we end up with the same formula since the action is symmetric with respect to q_1 and q_5 and this confirms that the delocalized charged black D-strings share the same phase structure as the charged black D5-branes. Second, when both q_1 and q_5 are non-zero, the reduced action is not the sum of the action due to D-strings and that due to D5-branes. This occurs not only in the first term (the internal energy multiplied by the common \bar{b} factor) but also in the second term (entropy) of the reduced action (32). However, for the corresponding extremal case (taking $f = 1$ in the action), the reduced action is indeed the sum of the above two corresponding extremal actions, as already indicated in footnote 12.

In other words, the dramatic change of the phase structure for either charged black

D5-branes or the delocalized charged black D-strings, when the other type of branes are added, are caused by the interaction between these two types of branes in the D1/D5 system considered. In comparison with the phase structures of black p-branes considered in [9], this change represents a phase structure change in characterization from the $\tilde{d} = 2$ case to $\tilde{d} > 2$ case. To be more precise, such a change is referred to any given continuous charge curve in the lower half of the charge square (including the diagonal line) in Fig. 1, starting on the q_5 -axis where $q_1 = 0$ (corresponding to $\tilde{d} = 2$), then in “lower region”, crossing the critical line and finally in “upper region” (when $q_1 \neq 0$, this curve appears to give rise to an effective $\tilde{d} > 2$).

The above discussion appears to imply that adding the delocalized charged D-strings to charged black D5-branes changes effectively its transverse dimensions \tilde{d} from $\tilde{d} = 2$ to a $\tilde{d} > 2$. This can be a possibility, given that the characteristic phase structure and critical phenomena for the charged black p-branes depends only on the dimensionality of transverse dimensions \tilde{d} . To confirm the possibility would require that the present D1/D5 system can be modeled effectively by a charged black brane system with an effective $\tilde{d} > 2$ but this will be beyond the scope of this paper. Instead, we will give a limited consideration by providing one piece of evidence in support of this connection.

At least for the charged black p-branes, the underlying phase structure is closely related to the corresponding critical point and the latter is completely determined by the transverse dimensions \tilde{d} , not the details of the underlying system. We also know that the charged black p-branes have a completely different phase structure for $\tilde{d} > 2$ and $\tilde{d} = 2$, and the corresponding critical parameters such as q_c and b_c show a pattern of decrease when \tilde{d} increases from $\tilde{d} = 2$ to $\tilde{d} = 7$ [9] as shown in the following table:

\tilde{d}	q_c	b_c
2	0.333333	0.288675
3	0.141626	0.199253
4	0.090672	0.159921
5	0.064944	0.134632
6	0.049599	0.116698
7	0.039529	0.103210

Table 1. The critical parameters q_c and b_c for black p-branes with $\tilde{d} \geq 2$

In other words, the decrease of both q_c and b_c corresponds to the increase of \tilde{d} . For the present D1/D5 system, when $q_1 = 0$, i.e., without the presence of the delocalized D-strings, this system is nothing but the above charged black 5-branes with $\tilde{d} = 2$. We now deform this charged black 5-branes by adding delocalized D-strings and we end up

with a richer phase structure, as discussed earlier, with a critical line described by a pair of critical charge (q_{1c}, q_{5c}) . The present critical parameters¹³ q_{1c}, q_{5c}, b_c for $2/3 \leq f_c \leq 1$ are given in Table 2.

f_c	q_{1c}	q_{5c}	b_c
1	0	0.333333	0.288675
0.96	0.000239	0.332405	0.288265
0.92	0.001431	0.329431	0.287193
0.88	0.004216	0.324028	0.286295
0.84	0.009350	0.315613	0.282938
0.80	0.017906	0.303256	0.279845
0.76	0.031669	0.285292	0.276135
0.72	0.054382	0.258084	0.271828
2/3	0.153093	0.153093	0.265165

Table 2. The present critical parameters q_{1c}, q_{5c} and b_c for $2/3 \leq f_c \leq 1$.

Note that $f_c = 1$ corresponds to the simple black D5-branes with $\tilde{d} = 2$ without the delocalized D1, i.e. $q_{1c} = 0$. The corresponding q_{5c} and b_c values are precise the ones given in the Table 1. Examining the pattern of the critical parameters, we see first that the critical b_c decreases in the same direction as f_c does. If such an effective description is assumed, the correlation between b_c and \tilde{d} given in Table 1 mentioned above implies an increase of the present effective \tilde{d} , from the original $\tilde{d} = 2$ to a new $\tilde{d} > 2$ for $f_c < 1$. One expects that the largest effective \tilde{d} corresponds to $f_c = 2/3$ and $b_c = 0.265165$ given in Table 2. As for $f_c < 1$, we have a non-vanishing q_{1c} and as we see from Table 2, the present q_{5c} value is no longer its original one $q_{5c} = 1/3$ (i.e., corresponding to $q_{1c} = 0$). This further implies that the new q_{5c} already takes the effect of the non-vanishing q_{1c} into consideration. So this new q_{5c} qualitatively appears as an effective q_c given in Table 1. Its decreasing pattern correlates with that of b_c lends further support to this assertion.

The largest q_{1c} corresponds to the symmetric point in Figure 1 where $q_{1c} = q_{5c} = 0.153093$ (or the exact value of $\sqrt{6}/16$) and $f_c = 2/3$. Here we have $b_c = 0.265165$ (or the exact value of $3\sqrt{2}/16$), still far away from $b_c = 0.199253$ of $\tilde{d} = 3$ given in Table 1. So we expect that the effective $\tilde{d} > 2$ but still far away from $\tilde{d} = 3$, if such an effective description exists indeed. So a brane charge has a potential in generating effectively a fraction of spatial dimension.

¹³As stated earlier, we have chosen the lower half of the critical line in Fig. 1 for the sake of comparing with the black D5-branes. Note that $q_{1c} = \sqrt{6}/16$ corresponds to the symmetric point.

4 Discussion and conclusion

In this paper, we explore means which can be used potentially to change qualitatively the phase structure of black p-branes in canonical ensemble. Following [9], we know that the charged black branes with their $\tilde{d} = 2$ serve as a borderline between the black branes with $\tilde{d} = 1$ and those with $\tilde{d} > 2$ in phase structure. So it should be much easier to realize such a change if we focus on the $\tilde{d} = 2$ branes, which in $D = 10$ are black D5 branes (or NS5-branes). The simplest system to consider is the black D1/D5. The delocalized charged black D-strings alone are found to have the exact same phase structure as the charged black D5-branes without a van-der Waals-Maxwell liquid-gas type structure. Given what we know about D1 and D5 in the BPS D1/D5 case, one would naively think that when the two are combined, the resulting phase structure should remain as before. However, it turns out that the resulting one is much richer and contains the liquid-gas type, with now a critical line instead of a critical point. We find that the physical reason for this to occur is the existence of a non-vanishing interaction between charged black D-strings and charged black D5-branes, which differs from that between the delocalized black D-strings or between black D5-branes.

The reduced D1 charge q_1 and the reduced D5 brane charge q_5 form the interior of a two-dimensional charge square. Given that the delocalized charged black D-strings and the charged black D5-branes each share the same phase structure, therefore so long the phase structure is concerned, we expect that there is a symmetry between q_1 and q_5 , as demonstrated in this paper. Therefore the fundamental region in the charge space is either of the two equivalent sub-regions so obtained by this symmetry, i.e., the region defined on either side of the diagonal line connecting the origin and its opposite corner of the square, plus the diagonal line excluding its two end points. In the paper, we choose for convenience the one nearby the q_5 -axis. There exists a critical line in this finite two-dimensional charge space with its one end approaching q_1 -axis and the other approaching q_5 -axis but neither of them reaching the corresponding axis. If we focus on the fundamental region, the corresponding critical line is the one with its one end terminating on the diagonal line and the other end approaching the q_5 -axis but not reaching it. Then this critical line separates the fundamental region into two parts. In the so-called “upper region” (defined earlier), any point in the charge space defines a unique thermodynamical stable black D1/D5 for given boundary data, similar to the $q > q_c$ case for the black p-branes with their $\tilde{d} > 2$. In the “lower region”, any point in the charge space defines a van der Waals-Maxwell liquid-gas type phase diagram for the black D1/D5 system for which there usually exist two local thermodynamical stable D1/D5 states, one small and

one large in horizon size, similar to the $q < q_c$ case for black p-branes with also their $\tilde{d} > 2$. The small one is analogous to the liquid phase while the large one to gas phase. Which one is more stable depends on the given boundary data and the lower cavity temperature favors the small one while higher temperature favors the large one when compared with the so-called transition temperature. For each given point in this region, there is a unique transition temperature determined by this point and at which the large and small black D1/D5 systems have the same free energy, therefore can co-exist and there is a first-order phase transition since any exchange between these two states involves a change of entropy.

For any charge curve in “lower region” terminating on the critical line, we will have a first-order phase transition line ending on a second-order phase transition point when the charge point (q_1, q_5) reaches its critical point (q_{1c}, q_{5c}) on the critical line. So any charge curve crossing the critical line in the fundamental region defined above will have a phase structure like that of black p-branes with their $\tilde{d} > 2$. This naturally leads to a speculation that the presence of q_1 can have a consequence, at least effectively, of increasing the original D5-brane $\tilde{d} = 2$ to a $\tilde{d} > 2$ slightly. So to certain extent, we see that a brane charge can effectively give rise to a fraction of a spatial dimension.

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Appendix

We in this appendix consider the general charged black D(p - 4)/Dp system for $4 \leq p \leq 6$. The corresponding black configuration in Euclidean signature is

$$\begin{aligned}
ds^2 &= G_{p-4}^{\frac{p-11}{8}} G_p^{\frac{p-7}{8}} \left(f dt^2 + \sum_{i=1}^{p-4} dx_i^2 \right) + G_{p-4}^{\frac{p-3}{8}} G_p^{\frac{p-7}{8}} \sum_{i=p-3}^p dx_i^2 \\
&\quad + G_{p-4}^{\frac{p-3}{8}} G_p^{\frac{p+1}{8}} \left(\frac{d\rho^2}{f} + \rho^2 d\Omega_{8-p}^2 \right) \\
A_{[p-3]} &= -ie^{\alpha(p-4)\phi_0/2} [\tanh \theta_{p-4} - (1 - G_{p-4}^{-1}) \coth \theta_{p-4}] dt \wedge dx^1 \wedge \dots \wedge dx^{p-4} \\
A_{[p+1]} &= -ie^{\alpha(p)\phi_0/2} [\tanh \theta_p - (1 - G_p^{-1}) \coth \theta_p] dt \wedge dx^1 \dots \wedge dx^p \\
e^{2(\phi-\phi_0)} &= G_{p-4}^{-\alpha(p-4)} G_p^{-\alpha(p)},
\end{aligned} \tag{43}$$

where the metric is in Einstein frame, each form field is obtained following [6, 9] in such a way that the form field vanishes at the horizon so that it is well defined in the local inertial frame, ϕ_0 is the asymptotic value for the dilaton and

$$G_{p-4,p} = 1 + \frac{\rho_0^{7-p} \sinh^2 \theta_{p-4,p}}{\rho^{7-p}}, \quad f = 1 - \frac{\rho_0^{7-p}}{\rho^{7-p}}. \quad (44)$$

The horizon is at $\rho = \rho_0$ with a curvature singularity behind at $\rho = 0$. In the above,

$$\alpha(p) = \begin{cases} \frac{p-3}{2} & \text{for Dp-brane,} \\ -\frac{p-3}{2} & \text{for NSp-brane.} \end{cases} \quad (45)$$

Once again, for convenience, we generalize the radial coordinate transformation (2) from ρ to r to the present case as

$$\rho^{7-p} = r^{7-p} - r_-^{7-p}, \quad \rho_0^{7-p} = r_+^{7-p} - r_-^{7-p}, \quad (46)$$

where $r_+^{7-p} \equiv \rho_0^{7-p} \cosh^2 \theta_p \geq r_-^{7-p} \equiv \rho_0^{7-p} \sinh^2 \theta_p$. In terms of the new radial coordinate r , the horizon is now at $r = r_+$ and the singularity is at $r = r_-$. The configuration (43) can now be expressed as

$$\begin{aligned} ds^2 &= G_{p-4}^{\frac{p-11}{8}} \Delta_-^{\frac{7-p}{8}} \left(\frac{\Delta_+}{\Delta_-} dt^2 + \sum_{i=1}^{p-4} dx_i^2 \right) + G_{p-4}^{\frac{p-3}{8}} \Delta_-^{\frac{7-p}{8}} \sum_{i=p-3}^p dx_i^2 \\ &\quad + G_{p-4}^{\frac{p-3}{8}} \Delta_-^{-\frac{p+1}{8}} \left(\frac{\Delta_-^{\frac{p-5}{7-p}}}{\Delta_+} dr^2 + r^2 \Delta_-^{\frac{2}{7-p}} d\Omega_{8-p}^2 \right) \\ A_{[p-3]} &= -ie^{\alpha(p-4)\phi_0/2} [\tanh \theta_{p-4} - (1 - G_{p-4}^{-1}) \coth \theta_{p-4}] dt \wedge dx^1 \wedge \dots \wedge dx^{p-4} \\ A_{[p+1]} &= -ie^{\alpha(p)\phi_0/2} [\tanh \theta_p - (1 - \Delta_-) \coth \theta_p] dt \wedge dx^1 \dots \wedge dx^p \\ e^{2(\phi-\phi_0)} &= G_{p-4}^{-\alpha(p-4)} \Delta_-^{\alpha(p)}, \end{aligned} \quad (47)$$

where now

$$G_{p-4} = 1 + \left(1 - \frac{\Delta_+}{\Delta_-} \right) \sinh^2 \theta_{p-4}, \quad G_p = \Delta_-^{-1}. \quad (48)$$

Here

$$\Delta_{\pm} = 1 - \frac{r_{\pm}^{7-p}}{r^{7-p}}. \quad (49)$$

In the above metric, the Euclidean time is periodic with a periodicity

$$\beta^* = \frac{4\pi r_+ \cosh \theta_{p-4}}{7-p} \left[1 - \left(\frac{r_-}{r_+} \right)^{7-p} \right]^{-\frac{5-p}{2(7-p)}}, \quad (50)$$

which is the inverse of temperature of the system at $r = \infty$. By the same token, we place this system in a cavity at $r = r_B$. As before, r_B is merely a coordinate radius, not the physical one which, from the metric, is defined as

$$\bar{r}_B = r_B G_{p-4}^{\frac{p-3}{16}} \Delta_-^{\frac{(p-3)^2}{16(7-p)}}. \quad (51)$$

The physical parameters

$$\bar{r}_{\pm} = r_{\pm} G_{p-4}^{\frac{p-3}{16}} \Delta_{\pm}^{\frac{(p-3)^2}{16(7-p)}}. \quad (52)$$

In terms of these physical quantities, Δ_{\pm} once again keep their respective forms unchanged

$$\Delta_{\pm} = 1 - \frac{r_{\pm}^{7-p}}{r^{7-p}} = 1 - \frac{\bar{r}_{\pm}^{7-p}}{\bar{r}^{7-p}}, \quad (53)$$

and the inverse of local temperature at \bar{r}_B is

$$\begin{aligned} \beta(\bar{r}_B) &\equiv \Delta_+^{1/2} G_{p-4}^{\frac{p-11}{16}} \Delta_-^{-\frac{p+1}{16}} \beta^*, \\ &= \frac{4\pi\bar{r}_+}{7-p} \left(\frac{\Delta_+}{\Delta_-}\right)^{1/2} \left(\frac{\bar{r}_+}{\bar{r}_B}\right)^{\frac{5-p}{2}} \left(1 - \frac{\Delta_+}{\Delta_-}\right)^{\frac{p-5}{2(7-p)}} \left(1 + \frac{1 - G_{p-4}^{-1}}{\frac{\Delta_-}{\Delta_+} - 1}\right)^{1/2}, \end{aligned} \quad (54)$$

where we have used the expression for β^* given in (50). The respective charge can be computed for this configuration, following

$$Q_p = \frac{i}{\sqrt{2}\kappa} \int_{S_{\infty}^{8-p}} e^{-\alpha(p)\phi} * F_{p+2}, \quad (55)$$

as

$$\begin{aligned} Q_{p-4} &= \frac{(7-p)\bar{V}_4\Omega_{8-p}}{\sqrt{2}\kappa} e^{-\alpha(p-4)\bar{\phi}/2} \bar{r}_B^{7-p} (1 - G_{p-4}^{-1})^{1/2} \left(1 - \frac{\Delta_+}{\Delta_-} G_{p-4}^{-1}\right)^{1/2}, \\ Q_p &= \frac{(7-p)\Omega_{8-p}}{\sqrt{2}\kappa} e^{-\alpha(p)\bar{\phi}/2} (\bar{r}_+ \bar{r}_-)^{\frac{7-p}{2}}, \end{aligned} \quad (56)$$

where we have expressed relevant quantities in terms of either the corresponding physical ones or the fixed values on the cavity. Note that all the above formulas, when set $p = 5$, reduce to those of the D1/D5 system as expected.

Following what we did for the D1/D5 system, we have the Euclidean action as

$$\begin{aligned} I_E &= \beta E - S \\ &= -\frac{\beta\bar{V}_p\Omega_{8-p}}{\kappa^2} \bar{r}_B^{7-p} \left[\frac{7-p}{2} \sqrt{\Delta_+\Delta_-} + \frac{9-p}{2} \sqrt{\frac{\Delta_+}{\Delta_-}} - \frac{7-p}{2} \sqrt{\frac{\Delta_+}{\Delta_-}} (1 - G_{p-4}^{-1}) - (8-p) \right] \\ &\quad - \frac{2\pi\bar{r}_+\bar{V}_p\Omega_{8-p}}{\kappa^2} \bar{r}_B^{7-p} \left(\frac{\bar{r}_+}{\bar{r}_B}\right)^{\frac{5-p}{2}} \left(1 - \frac{\Delta_+}{\Delta_-}\right)^{\frac{9-p}{2(7-p)}} \left(1 + \frac{1 - G_{p-4}^{-1}}{\frac{\Delta_-}{\Delta_+} - 1}\right)^{\frac{1}{2}}, \end{aligned} \quad (57)$$

which reduces to the D1/D5 one (12) when $p = 5$. From this action, one can read the corresponding internal energy E and entropy S , respectively. Similarly, we define the reduced charge

$$\begin{aligned}\tilde{Q}_{p-4} &\equiv \frac{\sqrt{2}\kappa Q_{p-4} e^{\alpha(p-4)\bar{\phi}/2}}{(7-p)\bar{V}_4\Omega_{8-p}\bar{r}_B^{7-p}} \\ &= (1 - G_{p-4}^{-1})^{1/2} \left(1 - \frac{\Delta_+}{\Delta_-} G_{p-4}^{-1}\right)^{1/2} < 1, \\ \tilde{Q}_p &\equiv \left(\frac{\sqrt{2}\kappa Q_p e^{\alpha(p)\bar{\phi}/2}}{(7-p)\Omega_{8-p}}\right)^{\frac{1}{7-p}} = \sqrt{\bar{r}_+ \bar{r}_-},\end{aligned}\tag{58}$$

where we have used (56) for Q_{p-4} and Q_p , respectively. From the second expression of above, we can express \bar{r}_- in terms of \bar{r}_+ and further we can solve G_{p-4} in terms of \bar{r}_+ from the first equation. We end up with

$$\begin{aligned}\bar{r}_- &= \frac{\tilde{Q}_p^2}{\bar{r}_+}, \\ 1 - G_{p-4}^{-1} &= \frac{1}{2} \left[\sqrt{\left(\frac{\Delta_-}{\Delta_+} - 1\right)^2 + 4\tilde{Q}_{p-4}^2 \frac{\Delta_-}{\Delta_+}} - \left(\frac{\Delta_-}{\Delta_+} - 1\right) \right].\end{aligned}\tag{59}$$

Since in canonical ensemble, $\beta, \bar{r}_B, \bar{V}_p, \tilde{Q}_{p-4}$ and \tilde{Q}_p are all fixed, the only variable in the action (57) is now the physical horizon size \bar{r}_+ . For simplicity, just like before, we define the following reduced action

$$\begin{aligned}\bar{I}_E &\equiv \frac{\kappa^2 I_E}{2\pi\bar{r}_B^{8-p}\bar{V}_p\Omega_{8-p}} \\ &= -\frac{\beta}{2\pi\bar{r}_B} \left[\frac{7-p}{2} \sqrt{\Delta_+ \Delta_-} + \frac{9-p}{2} \sqrt{\frac{\Delta_+}{\Delta_-}} - \frac{7-p}{2} \sqrt{\frac{\Delta_+}{\Delta_-}} (1 - G_{p-4}^{-1}) - (8-p) \right] \\ &\quad - \left(\frac{\bar{r}_+}{\bar{r}_B}\right)^{\frac{7-p}{2}} \left(1 - \frac{\Delta_+}{\Delta_-}\right)^{\frac{9-p}{2(7-p)}} \left(1 + \frac{1 - G_{p-4}^{-1}}{\frac{\Delta_-}{\Delta_+} - 1}\right)^{\frac{1}{2}},\end{aligned}\tag{60}$$

where we have used the action (57) and which reduces to the D1/D5 one (16) when $p = 5$. Once again for simplicity, we define

$$x = \left(\frac{\bar{r}_+}{\bar{r}_B}\right)^{7-p} < 1, \quad \bar{b} = \frac{\beta}{4\pi\bar{r}_B}, \quad q_p = \left(\frac{\tilde{Q}_p}{\bar{r}_B}\right)^{7-p} < x < 1, \quad q_{p-4} = \tilde{Q}_{p-4} < 1.\tag{61}$$

In terms of these reduced quantities, we have

$$\Delta_+ = 1 - \left(\frac{\bar{r}_+}{\bar{r}_B}\right)^{7-p} = 1 - x, \quad \Delta_- = 1 - \left(\frac{\bar{r}_-}{\bar{r}_B}\right)^{7-p} = 1 - \frac{q_p^2}{x},\tag{62}$$

where in the second expression, we have used the first equation for \bar{r}_- from (59). Then the reduced action (60) can be expressed as

$$\begin{aligned} \bar{I}_E = & -2\bar{b} \left[\frac{7-p}{2} \sqrt{\Delta_+ \Delta_-} + \frac{9-p}{2} \sqrt{\frac{\Delta_+}{\Delta_-}} - \frac{7-p}{2} \sqrt{\frac{\Delta_+}{\Delta_-}} (1 - G_{p-4}^{-1}) - (8-p) \right] \\ & -x^{1/2} \left(1 - \frac{\Delta_+}{\Delta_-} \right)^{\frac{9-p}{2(7-p)}} \left(1 + \frac{1 - G_{p-4}^{-1}}{\frac{\Delta_-}{\Delta_+} - 1} \right)^{\frac{1}{2}}, \end{aligned} \quad (63)$$

where we have used the second equation for $1 - G_{p-4}^{-1}$ in (59) and the expressions in (62) for Δ_{\pm} , respectively. From

$$\frac{d\bar{I}_E}{dx} = 0, \quad (64)$$

we have the equation of state

$$\bar{b} = b_{q_{p-4}, q_p}(\bar{x}), \quad (65)$$

where

$$b_{q_{p-4}, q_p}(x) = \frac{x^{1/2}}{7-p} \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} \left(1 - \frac{\Delta_+}{\Delta_-} \right)^{\frac{p-5}{2(7-p)}} \left(1 + \frac{1 - G_{p-4}^{-1}}{\frac{\Delta_-}{\Delta_+} - 1} \right)^{1/2}, \quad (66)$$

i.e., the inverse of local temperature (54) divided by $4\pi\bar{r}_B$. The above equation of state $\bar{b} = b_{q_{p-4}, q_p}(\bar{x})$ is just the thermal equilibrium condition of the underlying system with the cavity. As always, the behavior of function $b_{q_{p-4}, q_p}(x)$ determines the underlying phase structure of the system. Actually,

$$\left. \frac{d^2 \bar{I}_E}{dx^2} \right|_{x=\bar{x}} \sim -\frac{db_{q_{p-4}, q_p}(\bar{x})}{d\bar{x}}, \quad (67)$$

where \bar{x} is a solution of the equation of state given in (65). So this equation once again says that the underlying system has its free energy a local minimum at $x = \bar{x}$ if the slope of $b_{q_{p-4}, q_p}(x)$ is negative there, therefore giving its local stability in thermodynamics.

Let us examine the behavior of b_{q_{p-4}, q_p} a bit in detail. As expected, if we set $q_{p-4} = 0$, it should give the corresponding one $b_q(x)$ for the black p-branes in $D = 10$ [9] with $\tilde{d} = 7 - p$ (here $q = q_p$). One can check with a bit algebra that this is indeed true, i.e.,

$$\begin{aligned} b_{0, q_p}(x) &= \frac{x^{1/2}}{7-p} \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} \left(1 - \frac{\Delta_+}{\Delta_-} \right)^{\frac{p-5}{2(7-p)}} \\ &= \frac{x^{\frac{1}{7-p}}}{7-p} (1-x)^{1/2} \left(1 - \frac{q_p^2}{x^2} \right)^{\frac{p-5}{2(7-p)}} \left(1 - \frac{q_p^2}{x} \right)^{-\frac{1}{7-p}} = b_{q_p}(x), \end{aligned} \quad (68)$$

where we have used the explicit expressions of Δ_{\pm} in (62) and $1 - G_{p-4}^{-1} = 0$ when $q_{p-4} = 0$. Note that $q_p < x < 1$. Let us examine what happens if we set $q_p = 0$ in $b_{q_{p-4}, q_p}(x)$. We have now

$$b_{q_{p-4}, 0}(x) = \frac{x^{\frac{p-5}{2(7-p)}}}{7-p} (1-x)^{1/2} \left(\frac{x + \sqrt{x^2 + 4q_{p-4}^2(1-x)}}{2} \right)^{1/2}, \quad (69)$$

which appears quite different from the above $b_{0, q_p}(x)$. From what we learned for the D1/D5 case, we expect that the two should have the same form if a proper variable is chosen here. Our experience from the D1/D5 case suggests that a new variable y for $b_{q_{p-4}, 0}$ be chosen here as

$$y = \frac{x + \sqrt{x^2 + 4q_{p-4}^2(1-x)}}{2} < 1, \quad (70)$$

from which we can solve to give

$$x = \frac{y \left(1 - \frac{q_{p-4}^2}{y^2} \right)}{1 - \frac{q_{p-4}^2}{y}}, \quad 1-x = \frac{1-y}{1 - \frac{q_{p-4}^2}{y}}. \quad (71)$$

Note that here $0 < x < 1$ gives $q_{p-4} < y < 1$. With these relations, we have

$$b_{q_{p-4}, 0}(y) = \frac{y^{\frac{1}{7-p}}}{7-p} (1-y)^{1/2} \left(1 - \frac{q_{p-4}^2}{y^2} \right)^{\frac{p-5}{2(7-p)}} \left(1 - \frac{q_{p-4}^2}{y} \right)^{-\frac{1}{7-p}}, \quad (72)$$

which has the exact same functional form as $b_{0, q_p}(x)$ given in (68), as expected. Therefore the delocalized D(p - 4) branes alone have the same phase structure as the Dp branes. The non-symmetric appearance for the charges q_{p-4} and q_p in either the action (60) or the function $b_{q_{p-4}, q_p}(x)$ (66) is due to our bias choice of variable x more favorable to Dp-branes, as in the D1/D5 case. There should exist a variable f , just like the D1/D5 case, with which both q_{p-4} and q_p appears symmetric in either the action or the function b_{q_{p-4}, q_p} , given what we know about the phase nature of either type of branes. From our experience on the D1/D5 case, a good guess for the present case is

$$f = \frac{\Delta_+}{\Delta_-} = \frac{1-x}{1 - \frac{q_p^2}{x}} < 1, \quad (73)$$

which reduces to (30) when $p = 5$, i.e., the D1/D5 case. Note that $q_p < x < 1$ gives $0 < f < 1$, where $x = q_p$ corresponds to $f = 1$ while $x = 1$ to $f = 0$. We solve from the above action (73) for x and have

$$x = \frac{1-f + \sqrt{(1-f)^2 + 4q_p^2 f}}{2}. \quad (74)$$

In terms of variable f , we have the reduced action (63)

$$\begin{aligned} \bar{I}_E = & -2\bar{b} \left[\frac{(9-p)f^{\frac{1}{2}}}{2} + \frac{7-p}{2f^{\frac{1}{2}}} - (8-p) - \frac{\sqrt{(1-f)^2 + 4q_{p-4}^2 f} + \sqrt{(1-f)^2 + 4q_p^2 f}}{2f^{1/2}} \right] \\ & - (1-f)^{\frac{1}{7-p}} \left(\frac{1-f + \sqrt{(1-f)^2 + 4q_{p-4}^2 f}}{2} \right)^{\frac{1}{2}} \left(\frac{1-f + \sqrt{(1-f)^2 + 4q_p^2 f}}{2} \right)^{\frac{1}{2}}, \end{aligned} \quad (75)$$

and the function b_{q_{p-4}, q_p} (66)

$$b_{q_{p-4}, q_p}(f) = \frac{f^{\frac{1}{2}}(1-f)^{\frac{p-6}{7-p}}}{7-p} \left(\frac{1-f + \sqrt{(1-f)^2 + 4q_{p-4}^2 f}}{2} \right)^{\frac{1}{2}} \left(\frac{1-f + \sqrt{(1-f)^2 + 4q_p^2 f}}{2} \right)^{\frac{1}{2}}, \quad (76)$$

either of which is, as expected, symmetric with respect to q_{p-4} and q_p and reduces to its correspondence (32) or (33) when $p = 5$.

The behavior of $b_{q_{p-4}, q_p}(f)$ vs f will determine the underlying phase structure when both q_{p-4} and q_p are non-zero. Note that we have here $4 \leq p \leq 6$. For $p = 4$, we have

$$b_{q_0, q_4}(f \rightarrow 0) \rightarrow 0, \quad b_{q_0, q_4}(f \rightarrow 1) \rightarrow \infty, \quad (77)$$

whose characteristic behavior when $f \rightarrow 1$ is similar to that when $q_0 = 0$, i.e., the charged black 4-branes with $\tilde{d} = 3 > 2$. So we expect that the present phase structure, though richer and having details different, is essentially similar to that of charged black 4-branes given in [9]. To be precise, any given charge curve crossing the critical line in the two-dimensional charge square as in the D1/D5 case will give a phase structure similar to that of charged black 4-branes with $\tilde{d} = 3$ but in details more like that of black p-branes with $\tilde{d} > 3$. The detail analysis will be similar to the D1/D5 case and will not be repeated here. For $p = 6$, we have

$$b_{q_2, q_6}(f \rightarrow 0) \rightarrow 0, \quad b_{q_2, q_6}(f \rightarrow 1) \rightarrow q_2 q_6 \neq 0, \quad (78)$$

whose characteristic behavior when $f \rightarrow 1$ is like that of black 5-branes with $\tilde{d} = 2$. So adding the delocalized D2 branes to D6 branes modifies the D6 original phase structure with $\tilde{d} = 1$ [9] to the one similar to that of black D5 branes or effectively to that of black p-branes with $\tilde{d} > 1$ (Note that though the D5 one is richer, the D5 and D6 phase structures are essentially not much different from that of chargeless one as discussed in [9].). Once

again, the D2/D6 phase structure detail is different and more richer, for example, having a two-dimensional charge square and the detail analysis can be similarly performed as for the D1/D5 system and will not be given here.

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